Math Review

Summer 2017

*Topic 4*

4. Real valued functions and correspondences

4.1. Functions and correspondences

We have so far extensively used multiple definitions of a function. As we recall, functions are single-valued. For a function , every is mapped to one and only one point , the point Imagine a consumer with a *strictly quasiconcave* utility function (we define this later but for now take it to mean a nicely behaving function that gives us a unique answer when optimized) who behaves according to the *Utility Maximization expectation*. His/her market behavior is characterized by a demand function which we derived from the utility maximization process. This chosen bundle of consumption a single-valued function.

Often, however, we need to analyze behavior that is not uniquely determined. Consumers may have a place where they are indifferent between bundles at a given price for example. In this case, behavior is not single-valued.

A correspondence is thus a set-valued function from to

— for every is a subset of Y.

In this class, we will not spend a whole lot of time on correspondences, but know that they exist and will come up often. Think of the choice structure we learned about earlier, remember that it could assign multiple elements from one set to the other? That is a correspondence (or function in the case of one element assigned)!

Let’s start this session by practicing an example which will use some of the math skills we have down this far.

I have not worked through it, but it should be manageable, so I will work on it at the same as you all.

Find the Hessian of the equation

4.2. Homogeneous functions and Euler’s Theorem

Homogeneity in functions: The function is homogenous of degree if:

*Example*. The function is homogeneous of degree , can you tell why?

Denote:

If this is homogenous of degree k, then we have (from above):

Let’s determine what

Thus using our definition above, Thus, it is homogenous of degree k.

: Can you try with ? What is the degree of homogeneity for this function?

It should be homogenous of degree

You will learn that cost functions, denoted by that is input prices and output , are homogeneous of **degree 1 in** . Without you thinking too much into the details, how would you adapt the above definition to write this out mathematically?

, where , (note concerns with r only! (compare to demand example)

I’ve told you the result, but let’s double check with a given cost function. This cost function is from the June 2015 Micro prelim:

My initial fear in production was the scary functional forms. But let’s practice understanding what we are looking at. We have the cost function expressed in terms of input prices and output . As per the functional form, we have two prices, , and two outputs

: Show that this cost function is homogeneous of degree 1 in r.

: We want to show that .

(shown)

One more practice. Demand if often expressed as , that is as a function of both prices, p and wealth, w.

What degree of homogeneity holds for the following demand curve (in both prices and wealth)? This is a one-line answer!

Demand functions are said to be homogeneous of degree \_\_ in prices and wealth. What that gets at is that there is no “money illusion.” The same percentage increase in prices and wealth will lead to no change in consumption.

You should be able to check for the homogeneity of any given function, easy or hard. They often look harder than what they are. This is a prelim question. Try it out:

*You are the proud owner of a very expensive Mercedes-Benz. It cost you a lot of money to buy that car, so now you spend all your money on only two things that give you utility: driving your car around, at a cost of per mile driven, and buying rice to eat, which has a price of . Your total wealth for these two activities is denoted by .*

*Your indirect utility function (which is like a utility function but measured in a different way – you will learn more about this later), denoted by , is:*

*The indirect utility function should be homogeneous of some degree* ***k*** *in prices and*

*wealth. What is* ***k****?*

*Euler’s Theorem*. Let be a continuously differentiable function. The function *f* is homogenous of degree if and only if:

Or more simply:

This can be simply expressed in vector notations as:

Can we check whether the above cost function satisfies Euler’s Theorem?

Since the cost function is homogeneous of degree , we need:

Thus,

Let’s practice and check Euler’s theorem for the case of:

If we are taking the derivative with respect to an arbitrary point v, then:

Another result that you may want to know is the following: if is a continuously differentiable function that is homogenous of degree, then is homogenous of degree for all

What this is telling us is that for instance in our previous exercise, since : was homogeneous of degree , then is homogenous of degree .

Can we check whether this is true for this given case above?

Is ?- Yes.

You should be able to check Euler’s for all the different functions we tried today. Practicing now is always good, although you have plenty of time to get accustomed to this.

4.3. Concavity and quasi-concavity in functions

A summary of useful properties for multivariable functions are *(some are reminders of things we already touched on before*):

* The function is monotone if for all , if then
* The function is strictly monotone if for all , if then
* The function is concave if for all we have
* The function is convex if for all we have
* The function is quasiconcave if for all we have

imply that

* The function is quasiconvex if for all we have

imply that

As you see, a concave function by definition is quasiconcave (not the other way though). You will learn more about this, but we often require utility functions to be quasiconcave rather than the stronger assumption of concavity. This is more micro than math, but when dealing with preferences and utility, ordering matters more than actual numbers. We care whether is ranked higher than , not by how much insomuch. Quasiconcavity is a good enough assumption to represent this ordering between alternatives.

4.3.1 Hessian Matrix in the mix

Now that we have functions of several variables, the second derivative check for convexity/concavity is the Hessian Matrix. In this part, we determine how to check for concavity and convexity.

Theorem. Let be a twice continuously differentiable function. Then, *is concave* if and only if the Hessian is ***negative semidefinite*** for all. The function is *strictly concave* if and only if the Hessian is ***negative definite***. The function is *convex* if and only if the Hessian is ***positive semidefinite*** for all and *strictly convex* if and only if the Hessian is ***positive definite***.

We start by defining what a principal minor is:

Let be an square matrix.

We can find *principal submatrices* of by forming the first rows and columns, where

What isWrite it out.

Evidently,

In each case the *principal minor* is the determinant of the matrix. It is given by:

Determinant of denoted as  **or**  is just the number .

To find the determinant of **,** we need to take the following difference:

For visual thinkers:

**Solid: +**

**Dashed:**

For a matrix, we have to do a little more:

You have all the information you need. Can we write out the final expression to obtain the determinant?

Given an matrix , we have the following:

* The matrix is positive definite if and only if its principal minors are all greater than 0:
* The matrix is negative definite if and only if its principal minors alternate in sign with the odd order ones being negative and the even order ones being positive:
* The matrix is positive semidefinite if and only if its principal minors are all greater than or equal to 0:
* The matrix is negative semidefinite if and only if its principal minors alternate in sign with the odd order ones being less than or equal to zero and the even order ones being greater than or equal to 0:

Consider the following example: . Is this a concave function?

Recall two definitions:

1. *is concave* if and only if the Hessian is ***negative semidefinite*** for all.
2. An matrix is negative semidefinite if and only if its principal minors alternate in sign with the odd order ones being less than or equal to zero and the even order ones being greater than or equal to 0:

, thus

What is

, thus

Thus is concave.

The test for quasiconcacity and quasiconvexity relies on a *bordered Hessian matrix* which is a little more complicated (but totally manageable). I will refer you to Mas-Colell’s Mathematical Appendix p. 938-939 to more information.

You have a utility function given by . This is called a Cobb Douglas utility function with commodities and . Is this utility concave?

(*Just a note, we define commodities to be nonnegative.*)

, thus

What is

Thus is concave.

3.8. Implicit functions

Implicit Functions: Thus far, we have focused on functions whereby we have an endogenous variable , and exogenous variables, , on the right hand side: . These would be explicit functions.

In some cases, we have . The endogenous variable is an implicit function of the exogenous variables

*Example*.

If we were to rearrange and express it as an explicit function, we have:

However, if we have , we do not have a corresponding explicit function. Implicit functions are very popular in economics. For instance, think of a profit function for a firm given by , where is the production function, p is price, and c is cost per unit. In order to find a local maximum, we need to take the first derivative and set it equal to 0. is an implicit function with solution

The *Implicit Function Theorem* tells us when we are able to find an explicit function formula for . Let be a continuously differentiable point in the open ball, , around the point where the point satisfies:

and ,

Then, there exists a continuously differentiable function in the open ball around such that:

1. for all

And the MOST used result (in my opinion) is that:

1. For each :

Example. Consider the equation in the neighborhood of and Let’s apply this equation to what we know about implicit functions.

Condition 1:

Condition 2: We have the

Thus, we can apply the Implicit Function theorem to the point . There exists a continuously differentiable function in the open ball around such that we have:

*Example.* You have a utility function given by: . You will be exposed to the concept of marginal rate of substitution (MRS) of good for good given by: . Intuitively, the MRS is the rate at which a consumer is ready to give up one good in exchange for another good while maintaining the same level of utility. (*I will not further elaborate on this concept as you will cover this carefully in Consumer Theory).*

Applying the implicit function theorem above:

*:* Given a utility function, denoted by Find